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A topological characterisation of holomorphic parabolic germs in the plane

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Abstract

In [GP], Gambaudo and Pécou introduced the "linking property" to study the dynamics of germs of planar homeomorphims and provide a new proof of Naishul theorem. In this paper we prove that the negation of Gambaudo-Pécou property characterises the topological dynamics of holomorphic parabolic germs. As a consequence, a rotation set for germs of surface homeomorphisms around a fixed point can be defined, and it will turn out to be non trivial except for countably many conjugacy classes.

1 Introduction

Let $\mathcal{H}^+$ be the set of orientation preserving homeomorphisms of the plane that fix 0, and let $h \in \mathcal{H}^+$. We are interested in the dynamics of the germ of h at 0. Imagine one wants to evaluate the "amount of rotation" in a neighbourhood V of 0 by looking at the way the orbit of some point $x \in V$ rotates around 0. Then two kinds of difficulties can arise:

- if the orbit of x leaves V after a small number of iterations, then the behaviour of x is not significant with respect to the local dynamics;
- if the orbit of x tends to the fixed point 0, then the rotation of x around 0 is not significant either, because it is not invariant under a continuous change of coordinates.

These difficulties has lead Gambaudo and Pécou to the statement of the "linking property" (see [GP, Pé]) which demands that inside each neighbourhood of 0 there exist arbitrarily long segments of orbits starting and ending not too close to 0. In this paper we prove that the only germs that do not share the linking property are the contraction, dilatation and holomorphic parabolic germs. To be more precise, let us define the *short trip property*, which is the negation of Gambaudo-Pécou property, as follows.

Definition 1. Let $f \in \mathcal{H}^+$. We say that f satisfies the *short trip property* if there exists a neighbourhood V of the fixed point 0, such that for every

neighbourhood W of 0, there exists an integer $N_W > 0$ such that for every segment of orbit $(x, f(x), \dots, f^n(x))$ which is included in V, and whose endpoints $x, f^n(x)$ are outside W, the length n is less than N_W.

Two homeomorphisms $f_1, f_2 \in \mathcal{H}^+$ are said to be *locally topologically conjugate* if there exists a homeomorphism $\varphi \in \mathcal{H}^+$ such that the relation $f_2 = \varphi f_1 \varphi^{-1}$ holds on some neighbourhood of 0. We are interested in the local dynamics near the fixed point 0, thus we consider maps up to local conjugacy. Note that any local homeomorphism locally coincides with a homeomorphism defined on the whole plane, so that working with globally defined homeomorphisms is just a matter of convenience and does not alter the results (see [Ham] or [LR], chapitre 2). As a consequence, to prove that two homeomorphisms are locally topologically conjugate it suffices to construct the conjugacy on a neighbourhood of 0.

Definition 2. Let $f \in \mathcal{H}^+$, and identify the plane with the complex plane $\mathbb{C}$. We say that f is a *locally holomorphic parabolic homeomorphism* (or just *parabolic*) if f is holomorphic on some neighbourhood of 0, $f'(0)$ is a root of unity, and for every positive n the map f^n is not locally equal to the identity.

Note that the hypothesis on $f'(0)$ amounts to saying that the differential of f is a rational rotation, and then the last hypothesis is equivalent to saying that f is not locally topologically conjugate to its differential. According to Camacho version of Leau-Fatou theorem, if $f \in \mathcal{H}^+$ is parabolic, then f is locally topologically conjugate to some map

$$z \mapsto e^{2i\pi \frac{p}{q}} z(1 + z^{qr}), \qquad \text{with } \frac{p}{q} \in \mathbb{Q}, q, r \geq 1.$$

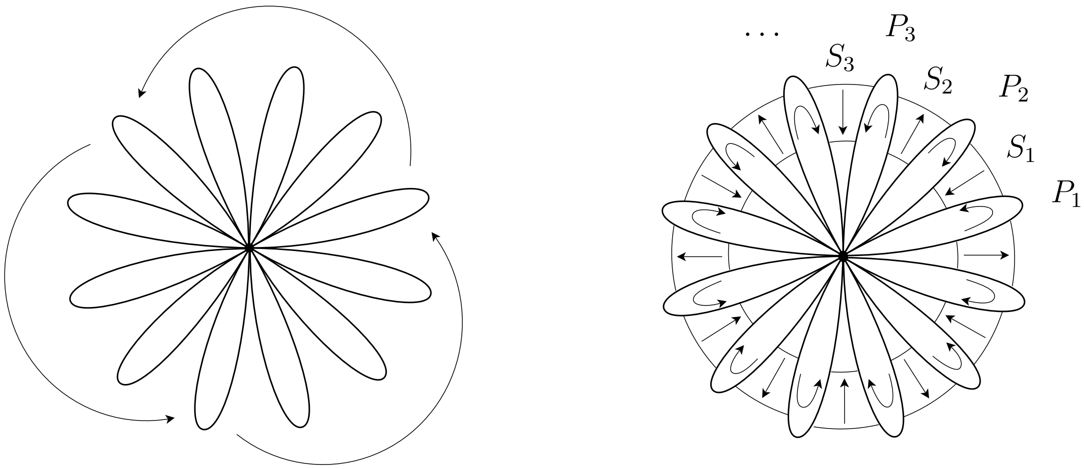
See [Cam], and figure 1.

We can now state our theorem.

Theorem 3. *Let f be an orientation preserving homeomorphism of the plane that fixes the point 0. Then f has the short trip property if and only if it is locally topologically conjugate to the contraction $z \mapsto \frac{1}{2}z$, to the dilatation $z \mapsto 2z$, or to a locally holomorphic parabolic homeomorphism.*

As a consequence there are only countably many conjugacy classes missing to satisfy Gambaudo-Pécou property.

In order to explain where theorem 3 comes from, let us first discuss Naishul theorem. In [GP] it was shown that Gambaudo-Pécou property holds when f preserves area, and then this property is used to prove Naishul theorem: *among area preserving homeomorphisms fixing 0 that are differentiable at 0 and whose differential is a rotation, the angle of the rotation is invariant under a local topological conjugacy.* Then the following nice generalisation of Naishul theorem is given by Gambaudo, Le Calvez and Pécou in [GLP]. As a generalisation of differentiability at 0, they consider the homeomorphisms f for which the fixed point can be "blown-up", *i.e.* replaced by an ideal circle in such a way that f can be extended to a circle



qr periodic petals for f action of f^q

Figure 1: Local topological dynamics of $f : z \mapsto e^{2i\pi\frac{p}{q}} z(1+z^{qr})$; here $q=3$, $p=1$, $r=2$, so that there is two orbits of attracting petals and two orbits of repulsive petals

homeomorphism (see the precise definition in [GLP]). They prove that for such homeomorphisms, *the Poincaré rotation number of the extended circle homeomorphism is invariant under a local topological conjugacy, unless f is a contraction or a dilatation.* The strategy of their proof is the following. If f has Gambaudo-Pécou property, then one can use the arguments in [GP]. Now assume that f is *indifferent*, that is, f admits arbitrarily small non-trivial invariant compact connected sets K containing 0; then one can use Caratheodory prime ends theory to associate a circle homeomorphism f_K to each such K, and use the rotation number of f_K to prove the topological invariance. Then one proves a last lemma asserting that *a germ which is not indifferent and does not have Gambaudo-Pécou property must be a contraction or a dilatation.*

As a consequence of Leau-Fatou theorem, parabolic maps are indifferent. Thus theorem 3 is a generalisation of this last lemma. Furthermore, it provides an alternative proof of the generalised Naishul theorem avoiding the use of prime-ends, as follows: we keep the arguments in [GP] to tackle homeomorphisms with Gambaudo-Pécou property; then, in view of theorem 3, it only remains to deal with parabolic homeomorphisms, for which the proof is easy because the local dynamics is completely understood.

More generally, in [LR2] we will define a local rotation set for any homeomorphism f in $\mathcal{H}^+$. This set is a subset of the line modulo integer translation, and it is a local topological conjugacy invariant. Then theorem 3 will entail that the local rotation set is non void as soon as f does not fall into the countably many conjugacy classes described by the theorem.

One can also think of theorem 3 as a local analogue of previous results showing that a simple dynamical property can imply a strong rigidity. The most striking result here is probably Hiraide-Lewowicz theorem that an expansive homeomorphism on a compact surface is conjugate to a pseudo-Anosov homeomorphism (see [Hi, Le]). Closer to our setting, Kérékjártó has shown that an orientation preserving homeomorphism of a closed orientable surface whose singular set is totally disconnected is topologically conjugate to a conformal transformation (see [BK, Ke34a, Ke34b]). Thus, for instance, an orientation preserving homeomorphism f of the plane is conjugate to a translation if and only if it has no fixed point and the family $(f^n)_{n\geq 0}$ is equicontinuous at each point for the spherical metric.

In some sense, theorem 3 highlights that it is easy to be locally conjugate to a locally parabolic homeomorphism: a homeomorphism that "looks like" a parabolic map will be conjugate to it. In contrast, the examples given in [BLR] reveal how difficult it is to be conjugate to the saddle homeomorphism $(2x, y/2)$, and in particular that it is not enough to preserve the hyperbolic foliation. A topological characterisation can be given, but it must take into account the sophisticated *oscillating set* (see the remark on fig. 3 in [BLR], as well as part III).

2 Dynamics of parabolic germs

Properties 12 and 13 below provide a first (classical) characterisation of parabolic germs in terms of *attracting* and *repulsive sectors* and *invariant petals*.

2.1 Contractions and attracting sectors

We begin by characterising the dynamics of contractions. Then we describe *attracting sectors*.Of course, similar results hold for dilatation and *repulsive sectors*, although we will not state them explicitly.

Let $f \in \mathcal{H}^+$. We will say that a sequence $(E_n)_{n\geq 0}$ of subsets of the plane *converges to* 0 if for every W neighbourhood of 0, all but finitely many terms of the sequence are included in W. The following result is very classical.

Proposition 4. *Let $f \in \mathcal{H}^+$. Let D be a topological closed disc*[1] *which is a neighbourhood of* 0*, and suppose that the orbit $(f^n(D))_{n\geq 0}$ converges to* 0*. Then f is locally topologically conjugate to the contraction $z \mapsto \frac{1}{2}z$.*

Proof. By hypothesis there exists $n > 0$ such that $f^n(D) \subset \mathrm{Int}(D)$. Choose some decreasing finite sequence of topological closed discs D_i with $D_0 = D$, $\mathrm{Int}(D_i) \supset D_{i+1}$, and $\mathrm{Int}(D_{n-1}) \supset f^n(D_0)$. Consider the set

$$O = \mathrm{Int}\left(D_{n-1}\right) \cap \mathrm{Int}\left(f(D_{n-2})\right) \cap \cdots \cap \mathrm{Int}\left(f^{n-1}(D_0)\right).$$

Let U be the connected component of O containing the fixed point 0. The hypotheses on the D_i's entail that $\mathrm{Clos}(f(O)) \subset O$. Since $\mathrm{Clos}(f(U))$ is

[1] A *topological closed disc* is a set homeomorphic to the closed unit disc.

connected and contains 0, we deduce that $\mathrm{Clos}(f(U)) \subset U$. Furthermore, according to a theorem of Kérékjártó, the set $D' = \mathrm{Clos}(U)$ is a closed topological disc (see [Ke23, LCY]). This disc satisfies $f(D') \subset \mathrm{Int}(D')$.

Now the annulus $D' \setminus \mathrm{Int}(f(D'))$ is a "fundamental domain" for f, and can be used to construct a local topological conjugacy between f and the contraction. $\square$

We will say that two sets S and S' *coincides in a neighbourhood of* 0, or *have the same germ at* 0, and we will write $S \overset{0}{=} S'$, if there exists a neighbourhood V of 0 such that $S \cap V = S' \cap V$.

Definition 5 (see figure 2)**.** An *attracting sector* is a topological closed disc S whose boundary contains 0, which coincides in a neighbourhood of 0 with its image $f(S)$, and whose orbit $(f^n(S))_{n\geq 0}$ converges to 0. The attracting sector is said to be *nice* if $f(S) \subset S$ and $S \setminus f(S)$ is connected. A *(nice) repulsive sector* is a (nice) attracting sector for f^{-1}.

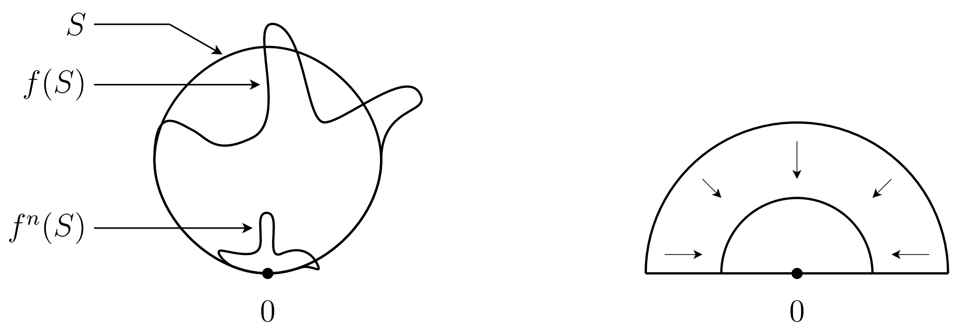


Figure 2: Attracting and nice attracting sectors

Claim 6.

1. *If S is an attracting sector then there exists a nice attracting sector S', included in S, and having the same germ as S at* 0.

2. *If S' is a nice attracting sector for f, then there exists a homeomorphism Φ between S' and the half-disc $S_0 = \{|z| \leq 1, y \geq 0\}$ such that the conjugacy relation $\Phi f = \frac{1}{2}\Phi$ between f and the contraction $z \mapsto \frac{1}{2}z$ holds on S'.*

Remark 7. Here are some easy consequences of item 2 of the claim.

1. The sets $\Phi^{-1}([-1, 0]), \Phi^{-1}([0, 1])$ are called the *sides* of the nice attracting sector; they do not depend on the choice of Φ.

2. There exists arbitrarily small nice attracting sectors within S'; moreover, any pair of points x, y on both sides of S' are the endpoints of the sides of some nice attracting sector included in S'.

3. Any homeomorphism Φ between the union of the sides of S' and the segment $[-1,1]$, satisfying the conjugacy relation $\Phi f = \frac{1}{2}\Phi$, can be extended to a homeomorphism between S' and S_0 conjugating f and $z \mapsto \frac{1}{2}z$ as in item 2 of the claim.

Proof of claim 6. Let S be an attracting sector. It is easy to see that there exists an arc α included in the boundary of S, whose interior[2] $\mathrm{Int}(\alpha)$ contains the fixed point 0, and such that $f(\alpha) \subset \mathrm{Int}(\alpha)$. Let us consider the set

$$A := \bigcup_{n \geq 0} f^{-n}(\alpha).$$

This set is clearly a continuous one-to-one image of the real line, and $f(A) = A$. By definition of an attracting sector, there exists an integer n_0 such that for every $n \geq n_0$, the set $f^n(S)$ is disjoint from the compact set $f^{-1}(\alpha) \setminus \mathrm{Int}(\alpha)$. Then one has $A \cap S = f^{-n_0}(\alpha) \cap S$. In particular, we can find a simple arc β such that $\alpha \cup \beta$ is a Jordan curve included in S, and whose intersection with A is reduced to α.

Let D_0 be the topological closed disc bounded by $\alpha \cup \beta$. Then D_0 is included in S and coincides with S in a neighbourhood of 0, and $D_0 \cap A = \alpha$. Note that for every n, $f^n(D_0) \cap A = f^n(D_0 \cap A) = f^n(\alpha)$. The disc D_0 is clearly an attracting sector. Let n_0 be a positive integer such that for every $n \geq n_0$, $f^n(D_0)$ does not meet β. Thus $f^n(D_0)$ is included in $\mathrm{Int}(D_0) \cup f^n(\alpha)$.

We can now find a closed topological disc $S' \subset D_0$, having the same germ at 0 as D_0, which is a nice attracting sector for f. For this we can make a construction similar to the proof of proposition 4, with the following adaptations. Now we choose the topological discs D_i's having the same germ at 0, containing α, and such that $\mathrm{Int}(D_i) \cup \{\alpha\} \supset D_{i+1}$ and $\mathrm{Int}(D_{n-1}) \cup \{f^n(\alpha)\} \supset f^n(D_0)$. The open set U is defined to be the unique connected component of O that has the same germ at 0 than the D_i's, and S' is the closure of U. Then one has $f(S') \subset U \cup f^n(\alpha)$, and $f^n(S')$ contains $f^n(\alpha)$; and thus $S' \setminus f(S')$ is connected. The details are left to the reader.

The proof of item 2 is straightforward by using the fundamental domain $\mathrm{Clos}(S' \setminus f(S'))$. □

2.2 Regular invariant petals

We first recall a theorem of Kérékjártó ([Ke34b]).

Theorem 8 (Kérékjártó)**.** *Let f be a homeomorphism of the plane that preserves the orientation, and suppose that for any compact set K the orbit $(f^n(K))_{n \geq 0}$ converges to the point at infinity in the sphere $\mathbb{R}^2 \cup \{\infty\}$. Then f is topologically conjugate to the translation $z \mapsto z+1$.*

Sketch of proof. By considering the space of orbits, the problem can be brought into the realm of the classification of surfaces: then it follows from the fact that any surface without boundary, whose fundamental group is the group of integers, is homeomorphic to the infinite cylinder (see for example [AS]). □

[2] The interior of a curve is defined to be the curve minus its endpoints.

Definition 9. An *invariant petal for f* is a topological closed disc P whose boundary contains the fixed point 0, and such that $f(P) = P$. An invariant petal is called *regular* if for every compact set $K \subset P \setminus \{0\}$, the sequence $(f^n(K))_{n\geq 0}$ converges to 0.

Remark 10. If P is a regular invariant petal then f has no fixed point on the topological line $\partial P \setminus \{0\}$. Thus we may endow this line with a *dynamical order* such that $f(x) > x$ for any point $x \neq 0$ on ∂P. The petal will be called *direct* if this dynamical order is compatible with the topological (usual) orientation of ∂P as a Jordan curve of the oriented plane (for which the interior of P is "on the left" of ∂P); in the opposite case it will be called *indirect*.

An adaptation of the proof of Kérékjártó theorem yields the following.

Claim 11. *Let P be a regular invariant petal for $f \in \mathcal{H}^+$. If P is direct then the restriction $f_{|P}$ is topologically conjugate,* via *an orientation preserving homeomorphism, to the restriction of the translation $z \mapsto z+1$ to the closed half-sphere $\{x + iy, y \geq 0\} \cup \{\infty\}$ of the Riemann sphere $\hat{\mathbb{C}}$. The same is true if P is indirect with $z + 1$ replaced by $z - 1$.*

2.3 Characterisation of parabolic homeomorphisms

We can now characterise the local dynamics of parabolic homeomorphisms. For any set D the *maximal invariant set* of D is the set $\cap_{n\in\mathbb{Z}} f^n(D)$ of points whose whole orbits are included in D.

Proposition 12 (see figure 1)**.** *Let $f \in \mathcal{H}^+$. Fix some integer $\ell \geq 1$. Then f is locally topologically conjugate to $z \mapsto z(1+z^\ell)$ if and only if there exists a neighbourhood of 0 which is a topological closed disc D called a* nice disc*, such that*

1. *the maximal invariant set of D is the union of 2ℓ regular invariant petals $P_1, \dots, P_{2\ell}$, whose pairwise intersections are reduced to $\{0\}$;*
2. *the sets $\partial D \cap P_i$ are connected, and the cyclic order of these sets along ∂D coincides with the order of the indices $i \in \mathbb{Z}/2\ell\mathbb{Z}$;*
3. *for every i, let S_i be the closure of the connected component of $D \setminus (P_1 \cup \dots \cup P_{2\ell})$ meeting both P_i and P_{i+1}, then S_i is a nice attracting sector for odd i and a nice repulsive sector for even i.*

The next statement takes into account a possible permutation of the petals.

Proposition 13. *Let $f \in \mathcal{H}^+$ and suppose that for some positive integer n_0 the map f^{n_0} is locally topologically conjugate to a parabolic homeomorphism. Then so is f.*

The proofs are delayed until section 4.

3 Proof of the theorem

From now on, f denotes an orientation preserving homeomorphism of the plane that fixes 0 and satisfies the short trip property. We fix some open neighbourhood V of 0 as in the definition of the short trip property, and define the sets

$$W^s(V) = \bigcap_{n \geq 0} f^{-n}(V) \quad \text{and} \quad W^u(V) = \bigcap_{n \leq 0} f^{-n}(V).$$

Since our hypothesis is symmetric in time both sets share the same properties, and we will usually restrict the study to $W^s(V)$.

3.1 Orbits

The following lemma shows in particular that the orbits of points near 0 can only converge to 0 or escape from the neighbourhood V. Note that this lemma still holds in any dimension.

Lemma 14.

1. *For every compact subset K of $W^s(V) \setminus \{0\}$, the sequence $(f^n(K))_{n \geq 0}$ converges to 0.*
2. *The set $W^s(V) \setminus \{0\}$ is open.*
3. *The set $W^s(V) \cup W^u(V)$ is a neighbourhood of 0.*
4. *If $W^s(V)$ is a neighbourhood of 0 then f is locally topologically conjugate to $z \mapsto \frac{1}{2}z$; if $W^u(V)$ is a neighbourhood of 0 then f is locally topologically conjugate to $z \mapsto 2z$.*

Proof. Let K be a compact subset of $W^s(V) \setminus \{0\}$, and let W be some neighbourhood of 0 disjoint from K. Note that by definition of $W^s(V)$, for every positive n, $f^n(K) \subset V$. Now let N_W be given by the short trip property. Then the property forces $f^n(K) \subset W$ for every $n > N_W$. This proves item 1 of the lemma.

Let $x \neq 0$ be some point in $W^s(V)$, and W a neighbourhood of 0 whose closure does not contain x, and such that $W \cup f(W) \subset V$. Let N_W be given by the short trip property. Let

$$O = \left(\bigcap_{n=0}^{N_W} f^{-n}(V) \right) \setminus \mathrm{Clos}(W).$$

This is an open set that contains x. We prove item 2 of the lemma by showing that O is included in $W^s(V)$. To see this, let $y \in O$. By definition of N_W in the short trip property we have $f^{N_W}(y) \in W$. Then we claim that $f^n(y) \in W$ for every $n \geq W$, which will entail $y \in W^s(V)$ as wanted. Assume by contradiction that our claim does not hold and let n_0 be the least integer after N_W such that $f^{n_0}(y) \notin W$. Since $f(W) \subset V$ we have

$f^{n_0}(y) \in V$. The segment of orbit $y, \dots, f^{n_0}(y)$ contradicts the definition of N_W. This completes the proof of item 2.

We consider again a neighbourhood W of 0 such that $W \cup f(W) \cup f^{-1}(W) \subset V$ and N_W given by the short trip property. We define the following neighbourhood of 0,

$$Z = \bigcap_{n=-N_W}^{N_W} f^{-n}(W).$$

We prove by contradiction that $Z \subset W^s(V) \cup W^u(V)$. Assume some point $x \in Z$ does not belong to $W^s(V)$ nor to $W^u(V)$. Since $W \subset V$, the orbit $(f^n(x))$ of x leaves W both in the past and in the future; but by definition of Z this cannot happen for n between $-N_W$ and N_W. Let r, s be the least positive integers such that the points $f^{-r}(x)$ and $f^s(x)$ do not belong to W; since $f(W) \cup f^{-1}(W) \subset V$ both points belong to $V \setminus W$ and again we have found a segment of orbit of length $r + s > 2N_W$ contradicting the definition of N_W.

Finally we notice that item 4 is a consequence of item 1 and the topological characterisation of contractions (proposition 4 above). □

3.2 Construction of the petals

We still consider a homeomorphism $f \in \mathcal{H}^+$ satisfying the short trip property, and from now on we assume that f is not locally conjugate to the contraction $z \mapsto \frac{1}{2}z$ nor to the dilatation $z \mapsto 2z$. We aim to prove that f is locally conjugate to a parabolic homeomorphism by ultimately applying propositions 12 and 13. The main task will be to contruct the family of periodic petals. As a first approximation we will select a finite number of connected components of $W^s(V) \cap W^u(V) \setminus \{0\}$, hoping to find one petal inside each of these components.

We fix an open neighbourhood V of 0 as before, and we assume V is simply connected. According to item 3 of the previous lemma, we can choose a topological closed disc D which is a neighbourhood of 0 and included in $W^s(V) \cup W^u(V)$. According to item 4, since we excluded the cases of contractions and dilatations, D is not included in $W^s(V)$ nor in $W^u(V)$. By compactness we can decompose ∂D as the concatenation of $2\ell \geq 2$ arcs $\alpha_1, \dots, \alpha_{2\ell}$ such that α_i is included in $W^s(V)$ for odd i and in $W^u(V)$ for even i. We make the following minimality hypothesis: *the number ℓ is minimal among all such choices of topological closed discs D and decompositions of ∂D.*

For every i (integer modulo 2ℓ) the common endpoint point x_i of α_{i-1} and α_i belongs to $W^s(V) \cap W^u(V)$. We denote by $\mathcal{C}_i$ the connected component of $W^s(V) \cap W^u(V) \setminus \{0\}$ that contains x_i. According to item 2 of lemma 14, this set $\mathcal{C}_i$ is open. Let D' be a topological closed disc; since V is simply connected, if $\partial D' \subset V$ then $D' \subset V$. Applying this to the iterates of D', we see that if $\partial D' \subset W^s(V) \cap W^u(V) \setminus \{0\}$, then $D' \subset W^s(V) \cap W^u(V)$. Since $W^s(V)$ is not a neighbourhood of 0, we get the following consequence.

Lemma 15. *Any connected component of $W^s(V) \cap W^u(V) \setminus \{0\}$ is open and simply connected. In particular, the sets $\mathcal{C}_i$ are homeomorphic to the plane.*

The next lemma is the fundamental step in the construction of the periodic petals. No dynamics is involved here; indeed, we will only need properties 2 and 3 from lemma 14 on the topology of $W^s(V)$ and $W^u(V)$.

Lemma 16. *For every i, the closure of $\mathcal{C}_i$ contains the fixed point* 0*.*

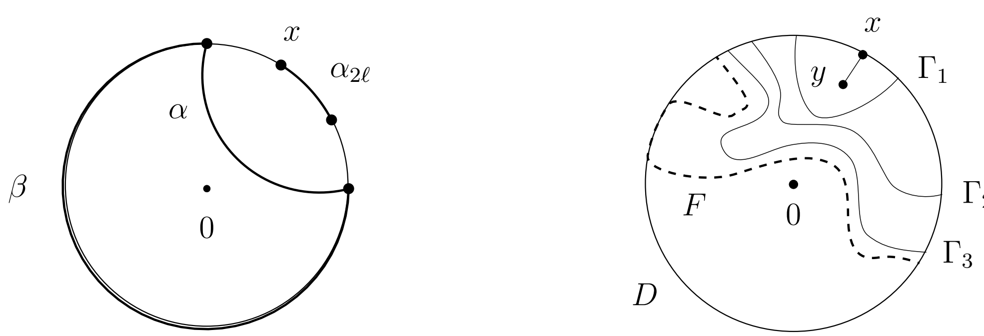
Proof. For notational simplicity we assume $i = 1$, and we note $\mathcal{C} = \mathcal{C}_1$ and $x = x_1 \in \alpha_{2\ell} \cap \alpha_1$. Using Schoenflies theorem, up to a change of coordinates, we can assume that D is a euclidean closed disc.

We will argue by contradiction. Assuming that 0 does not belong to the closure of $\mathcal{C}$, we will construct a simple arc α with the following properties (we denote by $\partial\alpha$ the set of endpoints of α and set $\mathrm{Int}(\alpha) = \alpha \setminus \partial\alpha$):

1. $\mathrm{Int}(\alpha) \subset \mathrm{Int}(D)$, $\partial\alpha \subset \partial D$;

2. α separates[3] x from 0 in D;

3. either $\alpha \subset W^s(V)$ and $\partial\alpha \cap W^u(V) = \emptyset$,
   or $\alpha \subset W^u(V)$ and $\partial\alpha \cap W^s(V) = \emptyset$.



(a) The arc α (b) The sequence Γ_k and the limit set F

Figure 3: Proof of lemma 16

From this we will get a contradiction as follows (see figure 3, (a)). Assume for example that the first case of the last item holds. Let $1 \le i_1 \le i_2 \le 2\ell$ be such that the endpoints of α are respectively included in α_{i_1} and α_{i_2}. Since $\partial\alpha$ does not meet $W^u(V)$, both i_1 and i_2 are odd, and in particular $1 \le i_1 \le i_2 < 2\ell$. Let $\beta \subset \partial D$ be the arc with the same endpoints as α and not containing x: then β is covered by $\alpha_{i_1} \cup \cdots \cup \alpha_{i_2}$, and from the second point we see that the Jordan curve $\alpha \cup \beta$ surrounds 0. Since $\alpha \cup \alpha_{i_1} \cup \alpha_{i_2}$

[3] A set A *separates* two points in a set B if the two points belong to distinct connected components of $B \setminus A$.

is included in $W^s(V)$, we can write $\alpha \cup \beta$ as the concatenation of $i_2 - i_1$ arcs, each included in $W^s(V)$ or $W^u(V)$. This contradicts the minimality hypothesis on ℓ since $i_2 - i_1 < 2\ell$.

We now assume that $0 \notin \mathrm{Clos}(\mathcal{C})$ and turn to the construction of such an arc α. According to lemma 15, there exists a homeomorphism $\Phi : \mathbb{R}^2 \to \mathcal{C}$. Let (D_k) be the sequence of images under Φ of the concentric discs with radius k and centre $\Phi^{-1}(x)$. Thus:

- $x \in D_1$,
- $D_k \subset \mathrm{Int}(D_{k+1})$,
- $\cup_{k \geq 0} D_k = \mathcal{C}$.

Let y be some point in $\mathrm{Int}(D)$ sufficiently near x so that the segment $[xy]$ is included in $D_1 \cap D$. For every k, the set $\partial D_k \cap \mathrm{Int}(D)$ is closed in $\mathrm{Int}(D)$ and separates y from 0 in $\mathrm{Int}(D)$ since $0 \notin D_k$. According to theorem V.14.3 in [New], there exists a connected component of $\partial D_k \cap \mathrm{Int}(D)$ that also separates y from 0. Let Γ_k denotes the closure of this component; thus Γ_k is a sub-arc of the Jordan curve ∂D_k with $\mathrm{Int}(\Gamma_k) \subset \mathrm{Int}(D)$, $\partial\Gamma_k \subset \partial D$, it separates x from 0 in D: in other words it satisfies the first two above properties required for the arc α.

Remember that the space of compact connected subsets of D is compact under the Hausdorff metric. Thus, up to extraction, we can assume that the sequence (Γ_k) converges to a compact connected set $F \subset D$ (see figure 3, (b)). Since $\Gamma_k \subset \partial D_k$, the set F is included in $\partial\mathcal{C}$. By assumption $\partial\mathcal{C}$ does not contain 0, so nor does the set F. Then again F separates y from 0 in $\mathrm{Int}(D)$: if not, there would exist an arc in $\mathrm{Int}(D)$ from y to 0 missing F, but then this arc would also miss Γ_k for sufficiently big k, contrary to the property that Γ_k separates y from 0.

Since $\mathcal{C}$ is a connected component of the open set $W^s(V) \cap W^u(V) \setminus \{0\}$, its boundary $\partial\mathcal{C}$ is disjoint from this set. Thus F is disjoint from $W^s(V) \cap W^u(V)$; and since it is included in D it is covered by the two open sets $W^s(V) \setminus \{0\}$ and $W^u(V) \setminus \{0\}$. Since F is connected it must be included in one of these two sets, and disjoint from the other one.

To fix ideas suppose that $F \subset W^s(V) \setminus W^u(V)$. Then for k large enough the arc Γ_k is also included in $W^s(V)$. Now this arc almost satisfies the three above properties required for the arc α, it only fails to have its endpoints outside $W^u(V)$. To remedy this we notice that, up to extraction, the two sequences of endpoints $(\Gamma_k(0))$, $(\Gamma_k(1))$ converges to some points in $z_0, z_1 \in F \cap \partial D$. Since $F \subset W^s(V) \setminus \{0\}$ we can choose $\varepsilon > 0$ so that the euclidean balls B_0, B_1 of radius ε and respective centres z_0, z_1 are included in $W^s(V)$. For k large enough Γ_k meets both balls, and then we construct the wanted arc α by modifying Γ_k near its endpoints: we replace two small extreme sub-arcs of Γ_k, respectively included in B_0 and B_1, by two euclidean segments reaching the points z_0 and z_1. Note that since D is a euclidean disc both segments, apart from their endpoints z_0, z_1, are included in $\mathrm{Int}(D)$. The endpoints z_0, z_1 of the resulting arc α are in F, thus outside $W^u(V)$, and α

fulfils the third property while still satisfying the first two. As we explained at the beginning of the proof, the existence of α contradicts the minimality of ℓ. $\square$

3.3 Periodicity of the petals

Unfortunately, we are not able to prove directly that the sets $\mathcal{C}_i$ of the previous section are periodic for f. To overcome this difficulty we will consider slightly larger sets $\mathcal{C}'_i$ which will turn out to be periodic. In the next section we will find a periodic petal inside each set $\mathcal{C}'_i$.

We suppose that the closure of V is included in some neighbourhood V' of 0 which still satisfies the short trip property. In other words, we apply the results of the previous sections with V small enough to meet this new assumption. We note that lemmas 14 and 15 apply to V'. Obviously the inclusions $W^s(V) \subset W^s(V')$ and $W^u(V) \subset W^u(V')$ hold. Let the disc D and the sets $\mathcal{C}_i$ be defined from V as in the previous section. Each set $\mathcal{C}_i$ is connected and included in $W^s(V') \cap W^u(V') \setminus \{0\}$, and thus it is included in one connected component of $W^s(V') \cap W^u(V') \setminus \{0\}$ which we denote by $\mathcal{C}'_i$.

Lemma 17. *The sets $\mathcal{C}'_i$ are periodic: for every i there exists some positive integer q_i such that $f^{q_i}(\mathcal{C}'_i) = \mathcal{C}'_i$.*

Proof. We first note that the set $W^s(V') \cap W^u(V') \setminus \{0\}$ is invariant under f, and hence for every n the set $f^n(\mathcal{C}'_i)$ is a connected component of $W^s(V') \cap W^u(V') \setminus \{0\}$.

We claim that for every i there exist infinitely many n such that $f^n(\mathcal{C}_i)$ meets the circle ∂D. Assuming the claim, we choose some limit point $x \in \partial D$ of the sequence $(f^n(\mathcal{C}_i))_{n \in \mathbb{Z}}$. Since the point x is a limit of points whose whole orbits are included in V, its orbit is included in $\mathrm{Clos}(V) \subset V'$: in other words x belongs to $W^s(V') \cap W^u(V') \setminus \{0\}$. Let O be the connected component of this last set containing x. According to lemma 15, O is open, and thus there exists infinitely many integers n such that $f^n(\mathcal{C}_i)$ meets O. For every such integer n, the set $f^n(\mathcal{C}'_i)$ is a connected component of $W^s(V') \cap W^u(V') \setminus \{0\}$ that meets O, thus it coincides with O. Thus we find two integers $n_1 < n_2$ such that $f^{n_1}(\mathcal{C}'_i) = f^{n_2}(\mathcal{C}'_i)$, which proves that $\mathcal{C}'_i$ is periodic.

We prove the claim. By lemma 16 the fixed point 0 belongs to the closure of $\mathcal{C}_i$. Since $\mathcal{C}_i \cup \{0\}$ is not a neighbourhood of 0, this point also belongs to the closure of $\partial \mathcal{C}_i$. Furthermore,

$$\begin{aligned}(\partial \mathcal{C}_i) \setminus \{0\} &\subset \partial \left(W^s(V) \cap W^u(V)\right) \setminus \{0\} \\ &\subset \left(W^s(\mathrm{Clos}V) \cap W^u(\mathrm{Clos}V)\right) \setminus \left(W^s(V) \cap W^u(V)\right).\end{aligned}$$

Consequently for any $z \in \partial \mathcal{C}_i \setminus \{0\}$ there exists an integer n such that $f^n(z) \in \partial V$. Let (z_k) be a sequence in $\partial \mathcal{C}_i$ converging to 0, then any sequence n_k such that $f^{n_k}(z_k) \in \partial V$ is unbounded, because the union of finitely many iterates of ∂V is a closed set which does not contain 0. For any k the set $f^{n_k}(\mathcal{C}_i)$ is connected, its closure contains 0 and meets ∂V, thus it also meets ∂D. This completes the proof of the claim. $\square$

3.4 Construction of the local conjugacy

We finally define the petals. According to the previous lemma we can choose some $n_0 > 0$ such that $F = f^{n_0}$ leaves invariant every set $\mathcal{C}'_i$. In view of proposition 13, theorem 3 will follow from the fact that F is locally conjugate to a locally holomorphic parabolic homeomorphism. Let us prove this fact.

Recall that $\mathcal{C}'_i$ is homeomorphic to the plane (lemma 15), and for any compact set $K \subset \mathcal{C}'_i$, the sequence $(f^n(K))_{n\geq 0}$ converges to 0 (lemma 14). Consequently theorem 8 tells us that for every i the restriction of F to the invariant set $\mathcal{C}'_i$ is conjugate to the plane translation $z \mapsto z+1$. Picking a horizontal line and bringing it back under the conjugacy, we see that the point x_i of $\mathcal{C}'_i$ is on a topological line $\Delta_i \subset \mathcal{C}'_i$ such that $F(\Delta_i) = \Delta_i$. Let P_i be the closed topological disc bounded by the curve $\Delta_i \cup \{0\}$ (here we use Schoenflies theorem). It is clear that P_i is a regular invariant petal for F.

The curve α_i, defined at the beginning of section 3.2, meets both petals P_i and P_{i+1}. Furthermore for odd i we have $\alpha_i \subset W^s(V)$ so by lemma 14 the sequence $(F^n(\alpha_i))_{n\geq 0}$ converges to 0, and the sequence $(F^n(\alpha_i))_{n\leq 0}$ converges to 0 for even i. Thus the construction of a local conjugacy between F and $z \mapsto z(1+z^\ell)$ now boils down to the following lemma.

Lemma 18 (see figure 1)**.** *Let $f \in \mathcal{H}^+$. Fix some integer $\ell \geq 1$. Assume the following hypotheses.*

1. *There exist 2ℓ regular invariant petals $P_1, \dots, P_{2\ell}$, whose pairwise intersections are reduced to $\{0\}$.*
2. *There exists a topological closed disc D which is a neighbourhood of 0, and whose boundary is the concatenation of 2ℓ arcs $\alpha_1, \dots, \alpha_{2\ell}$, each arc α_i having one endpoint on P_i and the other one on P_{i+1}.*
3. *For odd i the sequence $(f^n(\alpha_i))_{n\geq 0}$ converges to 0, and for even i the sequence $(f^n(\alpha_i))_{n\leq 0}$ converges to 0.*

Then f is locally conjugate to $z \mapsto z(1+z^\ell)$.

Note that we do not suppose that $\partial D \cap P_i$ is a connected set, nor that α_i does not meet some petal P_j with $j \neq i, i+1$, contrarily to proposition 12. An important step of the proof will be to check the not obvious fact that the petals indexation coincide with their cyclic order around 0.

Proof of lemma 18. Consider a homeomorphism $f \in \mathcal{H}^+$ satisfying the hypotheses of the lemma. The arc α_i contains a minimal sub-arc α'_i connecting P_i to P_{i+1}: the endpoints of α'_i are respectively on P_i and P_{i+1}, and its interior $\mathrm{Int}(\alpha'_i)$ is disjoint from P_i and P_{i+1}. Let i be odd, so that the sequence $(f^n(\alpha'_i))_{n\geq 0}$ converges to 0. Then we define an attracting sector S'_i as follows. We consider the curve obtained by concatenating the arc α'_i, the sub-arc of ∂P_i from the endpoint of α to 0 following the dynamical orientation of ∂P_i, and the similar sub-arc on P_{i+1} (see remark 10). This curve is clearly a Jordan cuve, it bounds the topological closed disc S'_i. It is not difficult to

see that S'_i is indeed an attracting sector. We apply item 1 of claim 6 to get a nice attracting sector $S_i \subset S'_i$ having the same germ as S'_i at 0. For even i we symmetrically define a repulsive sector S'_i and a nice repulsive sector S_i.

Since the petals are topological closed discs whose pairwise intersection is reduced to $\{0\}$, Schoenflies theorem can be used to prove that the union of the petals is homeomorphic to the model pictured on the left side of figure 1; but we still have to prove that their cyclic order is as shown on the right side of the figure (or the reverse one). For this we argue by contradiction. Suppose there exists some i such that the petals P_i and P_{i+1} are not adjacent: they are locally separated near 0 by the union of the other petals. Then there exists another petal P_j such that the sector S_i contains a neighbourhood of 0 in P_j; in other words the set $\mathrm{Clos}(P_j \setminus S_i)$ is a compact subset of $P_j \setminus \{0\}$. Using claim 11 that describes the dynamics of f on P_j, we find a point $x \neq 0$ whose full orbit $\{f^n(x), n \in \mathbb{Z}\}$ is included in $P_j \cap S_i$. But due to item 2 of claim 6 a nice attractive or repulsive sector contains no full orbit, which provides the contradiction.

Up to reversing the indexation, we may now assume that the petals are indexed in the positive cyclic order around 0 (so that Schonflies theorem provides an *orientation preserving* homeomorphism that sends each P_i on the model of figure 1). Note that the same argument as above, applied to P_i and P_{i+1} instead of P_j, shows that the interior of S_i is disjoint from P_i and P_{i+1}, and in particular the dynamical order on the boundaries of the petals is as indicated on figure 1: the petal P_i is direct for odd i and indirect for even i.

Up to replacing S_i with some smaller nice sector, we get that

1. for any i, j with $j \neq i, i+1$, we have $P_i \cap S_j = \{0\}$;
2. for any $i \neq j$, we have $S_i \cap S_j = \{0\} = f^{-1}(S_i) \cap S_j$.

Consider the set $D = P_1 \cup S_1 \cup \cdots \cup P_{2\ell} \cup S_{2\ell}$. Thanks to item 2 the maximal invariant set of D is the union of the petals P_i. Thus D is a topological closed disc satisfying the hypotheses of proposition 12. Now the lemma follows from the proposition. □

4 Proof of propositions 12 and 13

Proof of proposition 12. The fact that for the map $z \mapsto z(1+z^\ell)$ there exists a topological closed disc D satisfying properties 1,2,3 of the proposition is part of the proof of Camacho-Leau-Fatou theorem (see [Cam, Mil]).

We turn to the proof of the reverse implication. We consider a homeomorphism $f \in \mathcal{H}^+$ and a disc D satisfying properties 1,2,3 of the proposition. We have to prove that if $f' \in \mathcal{H}^+$ and a disc D' satisfies the same properties (with the same number ℓ) then f and f' are locally topologically conjugate. Note that the union of all sectors S_i and petals P_i is equal to D.

Let i be an odd integer. Since S_i is an attracting sector between P_i and P_{i+1}, the petal P_i is direct, while the petal P_{i+1} is indirect (see remark 10).

The same is true for f'. Thus according to claim 11, the restriction of f and f' to P_i and P'_i are conjugate. The conjugacies can be glued together to obtain an orientation preserving homeomorphism $\Phi : \cup P_i \to \cup P'_i$ which sends P_i onto P'_i and is a conjugacy between the restrictions of f and f'.

The image under Φ of $S_i \cap (P_i \cup P_{i+1})$ is not necessarily equal to $S'_i \cap \left(P'_i \cup P'_{i+1}\right)$. But using item 2 of claim 6 we can replace S_i and S'_i with smaller nice attracting sectors so that this equality becomes true (see item 2 of remark 7). We can now use item 3 of remark 7 to extend Φ to a homeomorphism between D and D', sending S_i onto S'_i and conjugating the restrictions of f and f'. We do this for every attracting or repulsive sectors S_i. We further extend Φ to a homeomorphism of the plane. The conjugacy relation $f'\Phi = \Phi f$ is satisfied on $D \cap f^{-1}(D)$. This completes the proof of the proposition. □

To prove proposition 13 we need a claim.

Claim 19. *Let Q_1, Q_2 be two invariant petals included in a regular invariant petal P for $F \in \mathcal{H}^+$. Then Q_1 meets Q_2, and there exists a unique connected component O of $\mathrm{Int}(Q_1) \cap \mathrm{Int}(Q_2)$ such that $F(O) = O$. Furthermore, the closure of O is a regular invariant petal for F.*

Proof. The first part is easily proved using the translation model given by claim 11. The only difficulty in the second part consists in checking that the closure of O is indeed a topological closed disc. But this follows from a previously quoted result of Kérékjártó ([Ke23, LCY]). □

Also note that if $Q \subset P$ are two regular invariant petals and P is direct then Q is direct.

Proof of proposition 13. Let $f^{n_0} = F$ be conjugate to a parabolic homeomorphism F_0. Up to increasing n_0, we can assume that $F'_0(0) = 1$, and thus F is locally conjugate to $z \mapsto z(1 + z^\ell)$ for some integer ℓ. Let D be a nice disc for F, and let $\{P_1, \dots, P_{2\ell}\}$ be the family of petals associated with D, as given by proposition 12.

For each i we choose a small invariant petal Q_i for F included in P_i. Since $f^{n_0} = F$ and Q_i is invariant for F, if Q_i is small enough then every iterates $f^n(Q_i)$ is included in D. Since $f^n(Q_i) \setminus \{0\}$ is connected and invariant for F, it is included in a connected component of the F-maximal invariant set of $D \setminus \{0\}$, that is, $f^n(Q_i)$ is included in some petal P_j. Let us fix j and consider the finite family of all the petals $f^n(Q_i)$ for $n \in \mathbb{Z}, i \in \mathbb{Z}/2\ell\mathbb{Z}$ which are included in P_j. We denote the intersection of their interiors by O_j. Applying claim 19 inductively we see that the closure of O_j is a regular invariant petal for F, let us call it $\bar{P}_j$.

By construction the petals in the family $\{\bar{P}_j\}$ are permuted by f, their pairwise intersections are reduced to 0, and their cyclic order around 0 is given by the cyclic order on the indices $i \in \mathbb{Z}/2\ell\mathbb{Z}$. Since f is an orientation preserving homeomorphism there exists i_0 such that for every i, $f(\bar{P}_i) = \bar{P}_{i+i_0}$. Furthermore, since f respects the dynamical orders induced by F on the boundary of the petals, i_0 must be even. The order n'_0 of the permutation

$i \mapsto i+i_0$ is a divisor of n_0 (maybe strict). It is easy to see that there exists another nice disc $\bar{D}$ for F whose maximal invariant set is the union of this family of petals. The nice attractive and repulsive sectors $\bar{S}_i$ for F, associated with $\bar{D}$, are clearly also attractive or repulsive sectors for $f^{n'_0}$, and according to claim 6 we can find within each $\bar{S}_i$ a nice attracting or repulsive sector $\bar{\bar{S}}_i$ for $f^{n'_0}$ having the same germ at 0. Now the topological closed disc $\bar{\bar{D}}$ defined as the union of all petals $\bar{P}_i$ and sectors $\bar{\bar{S}}_i$ is a nice disc for $f^{n'_0}$, the hypotheses of proposition 12 are satisfied, and $f^{n'_0}$ is conjugate to F.

Using these families of petals and sectors we are now in a position to construct a local conjugacy Φ between f and the model map $f_0 : z \mapsto e^{2i\pi\frac{i_0}{2\ell}}z(1+z^\ell)$. Note that $f_0^{n'_0}$ is conjugate to $z \mapsto z(1+z^\ell)$ and that f_0 permutes a family of regular invariant petals for $f_0^{n'_0}$. The construction of the conjugacy is similar to the one defined in the proof of proposition 12. Here is the main difference: since the petals are permuted by f, we have first to define a conjugacy Φ between $f^{n'_0}$ and $f_0^{n'_0}$ on some petal $\bar{P}_i$, and then there is a unique way to extend it to the f-orbit of $\bar{P}_i$ to get a conjugacy between f and f_0. We do the same for every f-orbit of petals, and for every f-orbit of sector. This completes the proof of proposition 13. $\square$

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